The Fairy-Tale Of The Lost Numbers

as the great-grandmother tells the great-grandson in 2025.

Editor's Preface

For the professional mathematician considering whether to tell this tale to his own children or to his university or grammar school students – older or young ones – the editor gives the following explanation:

The narrator uses the word 'number' in a restricted but strictly defined sense. She *calls numbers only those mathematical, calculable elements that* form a skew-body structure that is topologically coherent and locally compact.

To prevent a full-grown professor, who also loves surreal numbers, of feeling uncomfortable, let him always read 'Real Numbers' or 'Physical Numbers' instead of 'Numbers'.

It is important to the storyteller that with 'real numbers' one can also – in the traditional sense – differentiate and integrate.
She means: Newton needed his infinitesimal calculation for the physics of macrophysical movements; no modern physicist who wants to describe the world of microphysical quanta can do without infinity .

1. How Great-grandmother Sees Euclid's Elements

Great-grandmother was a strange woman. Unlike many old women who like their pots and pans, she loved the world of geometric shapes and mathematical numbers.

Great-grandmother told her great-grandson Noah a fairy-tale about numbers. This is how she began her story:

The God of Christians, Buddhists, Muslims and of all people and living beings on our globe and all other planets created numbers so that physicists, wherever they live, can perfectly describe nature by making successful use of the mathematical laws of nature. With the help of numbers, such spiritual beings as human beings cannot only look at the world but experience the world and its beauty on a spiritual level .
I want to remind you of an important example:
You already know the numbers that our mathematicians call "real numbers". We humans have learned to use these real numbers properly in physics. . It is only because we now understand how to describe physical movements exactly by means of these numbers that people can now, even though only since less than a hundred years, fly around the Earth in their self-made little moons, their "satellites"; or visit the moon. Since then, they have also been able to visit our neighbouring planet Mars in 2020.

However, it took human beings a few thousand years to gradually realize that numbers are the best and most beautiful of all tools to "precisely" grasp, admire and master the natural world into which they were born and which they perceive every day, with craftsmanship and philosophical thinking.

Well, it took humans even longer to realize that ALL the numbers that exist in the platonic heaven of the gods can already be discovered and used on earth.
And perhaps, great-grandmother interrupted herself, people have not yet finished their exciting voyage of discovery. Yes, great-grandmother continued thoughtfully, it was difficult and almost a little "scary" for people to explore that huge realm of numbers.

You can still see this today, dear Noah, in the fact that the numbers that were gradually discovered – and at first often only reluctantly accepted – still have very strange, sometimes almost "religious-tinged" names: Even today, mathematicians speak of "imaginary", "non-real", "irrational" and even "transcendental" numbers.
You already know, dear Noah, the circular number $\pi = 3.141592...$, don't you? Mathematicians say it is a "transcendent number".

That is why the scholars in Greece, who lived there around the year zero, still had a very hard time formulating that theory of the world quite simply and beautifully – comprehensible to everyone – which we call 'geometry' today.

Yes, geometry, great-grandmother was not afraid to say , is for me the first chapter of theoretical physics.
Euclid, for example, whom I see as one of the first great theoretical physicists, had great difficulty writing his 'Elements of Geometry' because he knew far fewer numbers than you have already encountered in school, and because he did not yet know anything at all about the famous – 'transcendental' – circle number π. Unfortunately, he neither knew the number "root out of two" that you are familiar with today. Nowadays , we humans have learned to use this number to describe the diagonal in the square.

The most astonishing thing is, great-grandmother concluded her first story-time, because her great-grandson looked really tired:
Euclid's contemporary philosophers were firmly convinced that such a completely absurd square root number could not and would never exist.

2. ISAAC, WILLIAM, ALBERT – Three Great Physicists

This is great-grandmother's second story-time.

I am still glad today that a second famous physicist – I will only call him by his first name here in my mathematical fairy-tale – was no longer so poor as far as the contents of his number-tool bag are concerned.

ISAAC was therefore able, albeit only 17 hundred years after EUCLID, to write that book on the physical nature with the help of numbers, to which he gave the strange title 'Principia Mathematica Philosophiae Naturalist'. Do you know this story, dear Noah?

With this famous book, we humans have learned that an apple falling from a tree, the sun rising in the morning and a spaceman walking on the moon obey the same laws precisely described by numbers.

You want to know how the fairy-tale of the lost numbers continues? Actually, this tragic story, great-grandmother sighed, only really begins now. Yes, this story that begins with WILLIAM is really tragic.

WILLIAM was born in the capital of Ireland in 1803. It is probably rightly said that even as a boy he spoke twelve languages – in addition to Greek and Latin. Thank God, he was not content with that. Why am I glad about that? Because before this theoretical physicist boldly went ahead, we humans did not yet know all the vocabulary of this language of numbers created by God. I mean, before WILLIAM, many pages in man's encyclopaedia of numbers were still empty. It was not until the middle of the 19th century that this researcher set out on a journey to finally discover all the numbers that exist.

However, even though WILLIAM already knew the new numbers a little, he still had to search for years for the laws on how to calculate with these numbers.

He found it particularly difficult to learn how to multiply these numbers, which were so strange to him at first. Today, we can better understand why it was so difficult for him to master even simple multiplications – the multiplication of these new, completely unfamiliar numbers, which WILLIAM later called "quaternions".
Does this name give you a vague idea of what was so difficult for WILLIAM and his contemporaries to understand?

As we know, we humans live and move in a natural world of only "three dimensions", because everyone can only move in three directions – forwards or backwards, to the right or left, upwards or downwards. There is no fourth movement independent of these three directions. WILLIAM had to learn, albeit reluctantly and laboriously, that at best non-human beings could picture these "four" numbers (quaternions); mythical beings or angels who - unlike us humans- could be at home not only in a three – but also four- dimensional world.

Although William was at home in many languages from his youth on, although as a young adult he learned to describe many aspects of nature with the help of those numbers commonly known at that time , once he had discovered his quaternions, he spent a whole twenty years of the rest of his life solely studying these new numbers in more detail. He also wanted to clearly demonstrate to his contemporaries the utmost importance of these newly discovered numbers.

But he failed. It is true that after his death, one of his students was still alive who appreciated these numbers as much as his teacher. Nevertheless, a few decades later, a strange development set in that still has repercussions to the present day. Many physicists after William's death could not believe that these "quite unimaginable", "four-dimensional" numbers could be meaningful and useful for a beautiful and clear description of physical nature.

Up to the turn of the century – around 1900 – there were still two powerfully eloquent parties among scientists: The first passionately took sides with the new, four-dimensional William numbers. But an ever-growing number of members of the – let's call them – "vector party" turned away in disgust from numbers they could not imagine. These vector supporters only wanted to have and use "numbers" that they thought only fit their three-dimensional natural world. Did these people, who borrowed a lot from William, divide all true numbers into two parts? Were they left with somewhat pathetic, three-dimensional pseudo-numbers, the vectors? "You can't even divide with these pseudo-numbers!" grumbled the members of the shrinking party of quaternion supporters. Soon the circle of these supporters became smaller and smaller . Their reputation and the reputation of the quaternions was almost completely ruined among the vector people.

Around 1900, however, the battle between quaternion friends and their opponents was still similar to the time of the Reformation, when Martin Luther, for example, had to fight against the traditionalists in Rome. In mathematical natural science, such fierce opposition had never been seen before. However, it ended differently from the struggle between the two religious traditions in the Reformation period. On the religious level, both parties, "Protestants" and "Catholics", remained powerful enough until today to successfully hold their ground against and alongside the other party. But to this day there are hardly any researchers in the exact mathematical sciences who consider these strange four-dimensional numbers as important as WILLIAM and his friends did back then.

This story is tragic, I believe, because this researcher died in 1865 and was no longer among the living in 1905. Had he still been alive at that time, the story might have been very different.

In 1905, a fourth great physicist began his epoch-making work. Today, many people know ALBERT because he discovered the famous identity of mass and energy, which has offered mankind today so many opportunities in economics and politics and caused great problems concerning nuclear energy.

It is less known that it was not until 1905, inspired by ALBERT's bold thinking, that more physicists dared to believe that there could be something "four-dimensional" in the natural world after all. Today, most physicists are even convinced that it is best to regard "time" as a quasi fourth dimension of our natural world.

What makes me sad, great-grandmother told me, is that most physicists today no longer know anything about the fact that the courageous discoverer of quaternions already knew how to understand the fourth dimension of these numbers as a time dimension. In other words, he already anticipated the hunches of mankind during his lifetime, which only became more credible to many physicists more than fifty years later through the work of ALBERT. William already understood his science of quaternions to be what he called the "Science of Pure Time", although he did not find any understanding for it during his lifetime. As I see it, on an intuitive level he was already closer to a conception of the fourth (time) dimension than is still common today. I believe that "time" is more than just a fourth appendage of our three space dimensions. Even today, great-grandmother confessed, I see reasons for the view that "time" shapes the spatial distance concept of physics, i.e. that it is more fundamental ("more real") than that.

3. The Fairy-tale of Great-grandfather's Triangle Numbers

In great-grandmother's family, there have been every now and then fathers and children, boys and girls, who have been interested in mathematics in addition to their jobs and daily duties. Similar to the way in which the love of music is reproduced in other families and always comes to the fore. Something similar was on Noah's mind when he asked his great-grandmother at the beginning of the story-time of the third evening:
So today quaternions are forgotten and lost numbers?

"Not quite," great-grandmother knew how to answer. She reported that until the turn of the millennium in 2000, they led an often unseen and unrecognized niche existence. But at the same time, there were always some scientists who wondered why the importance of these numbers had not yet been fully acknowledged in natural science. Despite the fact that the practical application of these four-dimensional numbers seemed to immediately suggest itself after it had become clear with ALBERT that the basic physical structure, which is nowadays called space-time, also has a four-dimensional shape.

And, what should also be noted in this context, great-grandmother reported: mathematicians were able to prove conclusively that no number-body with more than four dimensions exists.

Today, Noah, I want to tell you about the latest act of our number-fairy-tale-drama. This latest chapter of the story of the complete set of numbers has to do, dear Noah, with your great-grandfather KLAUS THEODOR.

Your great-grandfather proved in 1967:
The prevailing opinion since the discovery of the quaternionic numbers that these numbers are 'four-dimensional' per se; that they therefore cannot be visualized in the area of human perception and experience, is a prejudice.
Great-grandfather's essay "Triangles as Elements of Algebraic Bodies"[1] already showed that these numbers are not only imaginable as empty computational quantities but as TRIANGLES. As such, they can be sensually grasped by you and everyone in three dimensions.

However, it took another thirty years until great-grandfather showed how these WILLIAM numbers – and their multiplication! – could be illustrated more comprehensively .

But first, KLAUS THEODOR subjected Euclidean geometry, as you still learn it in school, to a modern "reformation". In two articles in 2001, he proved that one can measure the size of angles and justify the theory of angle measurements in triangles without the use of a metre scale.[2]

As the basic structures of a triangle – and thus of a quaternionic number – already exist in a three-dimensional space, the quaternions, as Klaus Theodor sees them, turned out to be basic figures in the space of our geometrical experience.[3] This does not necessarily have to be understood as a space created by Euclidean points. As long as one only measures angles

[1] See source (1) in chapter 4
[2] See source (2) in chapter 4
[3] See source (4) in chapter 4

in such a space without resorting to the measurement of length-like quantities – such as time and distance – these geometric elementary particles are *without any definite size*.
"Yes," great-grandmother suddenly pontificated quite excitedly, "they are neither large nor small. They are precisely defined in their shape by their three angles, but in other respects they behave in a similar way to physical quanta, which, as we know, also often have no well-defined size or location.[4]

Unfortunately, it has not yet become widely known in the community of physicists that the WILLIAM numbers, which originally could only be thought of as elements in an abstract four-dimensional space, can also be imagined as spherical triangles in the three-dimensional space of our perception and experience.

And a second, remarkable property of these numbers has not yet been noticed in the circle of natural scientists. Particularly for this reason, these calculation quantities, which WILLIAM considered to be very important and valuable numbers, are still like Cinderella, who has not yet been discovered in the fairy-tale. What has been overlooked about this number Cinderella so far?
I have already told you that WILLIAM, the discoverer of the quaternions, had great difficulty learning how to multiply these numbers. Why this was the case, I may explain here in my fairy-tale with the help of a parable: Even if you are a skilled calculator as a craftsman, you - as a recent immigrant to the USA – will not be able to bill your customer until you are familiar with the local currency and its unit, the dollar. The discoverer of the quaternions had to search a long time for such units, which were still unknown to him, before he could calculate correctly with these new numbers. Finally, such units – the basis of the quaternions and their multiplication rules – were discovered and everything was OK.

It was not until 1999, however, that it turned out that the idea the discoverer had formed of these numbers was a one-sided one . It turned out that in addition to – and on an equal footing with – the classical 'dollar currency' of numbers, there is a second complementary currency of quaternions. Just as an internationally active company can issue an invoice both in dollars and euros, there are two equally legitimate basic systems for quaternionic numbers. Only now has this second system been recognized as having equal rights within the framework of number representation.[5]

Only now can one convert between the two fundamental basic systems of numbers, just as one can convert between dollars and euros. Klaus Theodor, your great-grandfather, was also able to determine the conversion factor that is called the money rate in the monetary system.

The existence of two equal, complementary basic systems of numbers is interesting because earthly natural science has now for the first time a clear and beautiful tool to rethink and describe the *complementary* world of quantum physics.

Not only on modern quantum physics, but also on the theory of relativity founded by ALBERT – and on the connection between these two fundamental theories does this double structure of numbers throw a new light.

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[4] See source (5) in chapter 4
[5] See source (6) in chapter 4

The conversion formula developed by Klaus Theodor between the two base systems of numbers, the equation $\boldsymbol{i} = \boldsymbol{c} \cdot \boldsymbol{h}$ describes both: The world of relativity theory through the classical base of numbers and its measures $\boldsymbol{c} = \mathbf{1}$, $\boldsymbol{h} = \boldsymbol{i}$ and the world of quantum physics through the complementary base and its measures $\boldsymbol{c} = \boldsymbol{i}$, $\boldsymbol{h} = \mathbf{1}$.

I can understand, great-grandmother comforted her grandson Noah, that you do not yet quite understand why I find this equation $\boldsymbol{i} = \boldsymbol{c} \cdot \boldsymbol{h}$ mathematically and physically very exciting.

Your current lack of understanding is due to the fact that you have not yet fully grasped the meaning of the mathematical number sign **i** .

What does your great-grandfather – basically – want to say? "The sometimes real and sometimes imaginary number ***c*** describes the speed of light. And the sometimes real and sometimes imaginary number ***h*** describes the physical quantum of action. The measure ***c*** is real only in the range of validity of ALBERT's theories; the measure ***h*** is real only in the range of validity of quantum physics."

You think that's a bold hypothesis.

So do I.

Well, hypotheses can be bold or unbelievable, right or wrong, great-grandmother said.
But what I find bad, she went on, is that contemporary physicists do not want to acknowledge and understand such a statement and the equations associated with it; but rather find it "impossible" and scandalous. For many of your great-grandfather's results are purely mathematical. They can be understood and accepted by any physicist who thinks honestly like a mathematician.

Do "scholars" today, in 2025, still react like the philosophers of old? These also found unfamiliar numbers to be utterly nonsensical, 'irrational', merely "imaginary" and "impossible". Do they too react again exactly like the mathematicians and physicists of WILLIAM's time, who judged his new numbers to be practically useless and physically nonsensical?

Yes, the fairy-tale of the lost and forgotten numbers, great-grandmother concluded her fairy-tale a little sadly, will probably only come to a happy ending when the physicists who are still alive now will have died. Or if you, Noah, begin tomorrow to study great-grandfather's thoughts diligently and honestly. Here I have written down his original works for you:

4. About the Publications of the Great-grandfather

Great-grandmother is a strange woman. When she turned 100, her great-grandson Noah was already grown up. He visited her for her birthday. Even at her advanced age, she had not lost her passion for mathematical and theoretical questions of physics . You have also come, she smiled at her great-grandson, to consult the works of your great-grandfather KLAUS THEODOR . Today I am giving you the old originals, the "offprints", as they were called back then. But you can also find PDF copies of his works on the internet.

1. Let me tell you what I find particularly important about some individual works. Great-grandfather's work was published in 1967.

> **(1) Triangles as Elements of Algebraic Bodies, Klaus Ruthenberg, Praxis der Mathematik, Cologne 1967, p. 65-70.**

Perhaps, dear Noah, you should also study this article first, because in order to understand this work, you only need some elementary knowledge of vector calculus. You are already familiar with vector calculus from school or from the first courses before your bachelor's studies. You can read and understand this work easily because you will find nine drawings in this work that clearly illustrate everything. Great-grandfather already shows at an elementary level in this work: Euclid's well-known triangles can be understood as elements of a (skew) body. That is, they behave like the quaternions, the "four-dimensional" numbers of their discoverer WILLIAM.

2. In the first part of the article

> **(2) The Quaternionic Structure of 3-dimensional Natural Geometry, Klaus Ruthenberg, Journal Of Natural Geometry, 16, 1999, 125-140**

great-grandfather proves on the more abstract level of university mathematics, i.e. completely logically , that the multiplication of quaternions corresponds to the composition of Euclidean triangles; that the WILLIAM numbers are formally identical with our old familiar triangles. On page 132 you will find a list of seven peculiarities of this so really close relationship between Euclidean triangles and the quaternionic numbers.

For the first time, Grandfather also introduces the equation $\boldsymbol{c} \cdot \boldsymbol{h} = \boldsymbol{i}$.

He calls the complex(!) numbers **c, h** "fundamental metric constants".

The second part contains a first physical interpretation:
together with the special choice of $(\boldsymbol{c},\boldsymbol{h}) = (\mathbf{1},\boldsymbol{i})$ the derivation of basic kinematic and dynamic equations of the special theory of relativity works out.
Thus, a connection is established between physical areas where the speed of light **c** plays a fundamental role and those areas of physics where the quantum of action ***h*** dominates.
Great-grandfather concludes this second article with the theorem:
"In this way we can identify the spin ½ h of a physical particle with the Euclidean angle sum of each Hamiltonian quaternion. "

Great-grandmother urged: So, dear Noah, you should study your great-grandfather's later work in particular to see what reasons great-grandfather adduces for the fact that it can make a lot of theoretical sense to identify the two terms "physical quanta" and "mathematical numbers".

3. Do you, dear Noah, already have enough patience to understand in more detail which detours, and new paths great-grandfather had to take in order to arrive at his identification of quanta and numbers?
Already in the sixties of the twentieth century great-grandfather succeeded in looking at the Euclidean geometry that we are all "familiar with" from a "higher point of view". If you, Noah, as his great-grandson, are really interested in these both elementary and fundamental insights, I will gladly give you great-grandfather's "handwritten" copy of his manuscript.
"Elementary geometry as conformal invariant theory".

It was not until 1999-2001 that great-grandpa published two essays with parts of his findings in the London journal

(3) Journal of Natural Geometry, 19, 2001, 73 – 92, and 19, 2001, 93- 120.

The author was able to draw on the work of the mathematician Moebius on the so-called "conformal geometry". Only now, however, was it possible to justify the measurement of angles and their trigonometry in a purely conformal way, solely by means of double ratios, INDEPENDENT of Euclidean concepts such as "straight line", "circle", "length of line" and the like.

Perhaps I should also say here, great-grandmother remarked: Before these two works, it was nowhere clear to me that from the higher standpoint of this Moebius geometry, where one knows angles and their sizes, but not distances and their lengths, the Euclidean difference between circles and straight lines completely disappears. For this reason, great-grandfather still shied away from introducing a completely new concept . But I know that he would have preferred to speak of "Aden". Aden are geometric shapes like circles, which – like straight lines – have no centre; at the same time, "Aden" are like straight lines, which – like circles – are coherent. Traditionally, some people – perhaps only in Germany – speak of "conformal circles" instead of Aden; but then – for reasons of parity – one would also have to address an Ade as a "conformal straight line".
Yes, great-grandmother was a clever and at the same time headstrong woman. Before she interrupted her mathematical-physical chat with Noah for the celebratory birthday tea, she quietly blasphemed, because Noah was now an adult:
"Physicists can’t reconcile waves and particles, because they can’t reconcile straight lines and circles.”

4. Noah had red ears: Sometimes he found great-grandma's chatter quite demanding.

You know the pictures and explanations on the China poster on great-grandma's website www.natural-geometry.de, don't you?

So great-grandma asked for the birthday tea.
Here is Klaus Theodor's fourth article:

> **(4) Quaternions as Spherical Particles in 3-dimensional Space, Journal of Natural Geometry, 19, 2001, 121 – 138**

Let us read together how great-grandfather summarises the results of his article:

The triangle model of quaternions has its origin in conformal geometry ... The multiplication of quaternions is the composition of such spherical particles. Quaternions – not points or vectors – are the basic elements of natural space. Complex Hamiltonian numbers are pure angular entities. After defining a unit of length, one can use these numbers to describe the time and space components of physical events ...

The work published in the former USSR:

> **(5) Quanta perceived as Quaternions, Electromagnetic Phenomena, Kiev, Vol. 3, 1, (9), 2003**

represents in detail how we come to a simple, transparent understanding of the geometric-physical foundations of macrophysical and microphysical phenomena if, before local Euclidean structures are introduced, we assume an underlying structure in which not points, straight lines, circles, but Aden, DiAden = angles and TetraAden = TriAngle = quaternions are the first, fundamental geometric-physical elements of natural 3-dimensional space.

Many physicists have become accustomed – in Einstein's sense – to describing, for example, gravitational movements with differential geometry, i.e. by means of non-Euclidean geometry. Nevertheless, it remains difficult for everyone to imagine that the geometry in which we live, work and measure on a daily basis is no longer "Euclidean".
The elementary question is: what can a world look like, in which there are angles, but for example (for now) no straight lengths , thus no "large" and "small" of bodies. In such a natural world, each triangle = quaternion can be thought to be arbitrarily small or arbitrarily large. Such a figure has no "size", only a clearly defined angular structure.

Because quaternions have not been seen in this "size-free" way as fundamental elements of our natural space, as urelements of our natural geometry, while the normal "2-dimensional" complex numbers are known to every mathematician and physicist today and are familiar tools , great-grandfather first tries in his own work to represent the ordinary complex numbers "length-size-free" in such a way that he does not need the ordinary Gaussian/Argande plane and his Cartesian coordinate systems . His work on this, published in the USA, is entitled

> **(6) Angles Generate More Fundamental Frames Than Lengths, Hadronic Journal 26, 67 -92, 2003**

This work is of purely mathematical nature. It explains on this purely mathematical level that classical trigonometry on the higher level of the natural-conformal plane permits both a periodic and a non-periodic continuation. On page 81 the **Transformations of Duality** can be found**.**

Without these transformations, the two trigonometric systems are unrelated but equal in the real. The scalar unit i is the non-real operator between these two coordinate systems. Note the additional comments , the summary and the concluding "paradoxical" description of quaternionic "natural geometry". Perhaps, Noah, you too will ask yourself: is this geometry non-Euclidean, non-Cartesian?

On the website www.natural-geometry.de all original works can be found as PDF files under "Downloads".

To begin with, this website also shows the triangle representation of quaternions, together with their geometric multiplication, depicted on the "PEKING POSTER". In 2003, this poster introduced mathematicians in China to the new concept of quaternions as elements in the three-dimensional natural space of our experience.